Thermodynamics of charged anti-de Sitter black holes in canonical ensemble

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Abstract

As in the grand canonical treatment of Reissner - Nordström black holes in anti-de Sitter spacetime, the canonical ensemble formulation also shows that non-extremal black holes tend to have lower action than extremal ones. However, some small non-extremal black holes have higher action, leading to the possibility of transitions between non-extremal and extremal black holes.

Black hole thermodynamics continues to be an interesting field of study. A special area of recent interest is that of *extremal* black holes. While the entropy of ordinary (*non-extremal*) black holes has been taken to be a quarter of the horizon area for a long time, recently there has been some confusion in the case of *extremal* black holes. The semiclassical derivations of the entropy formula for non-extremal black holes do not directly apply to extremal black holes, and because of the difference in topology of euclidean extremal and non-extremal black holes, one cannot rely on extrapolation. In fact, euclidean studies indicate that extremal black holes should have zero entropy [1] even though the horizon area is nonzero. On the other hand, practical-minded people have tended to expect that extremal black holes should satisfy

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the area law just like non-extremal black holes from which they differ ever so slightly. One way of accommodating their point of view is to argue that there may be different ways of looking at extremal black holes. Usually, when one quantizes a classical theory, one tries to preserve the classical topology. In this spirit, one seeks to have a quantum theory of extremal black holes based exclusively on extremal topologies. As an alternative, one can have a quantization where a summation is carried out over topologies. Then, in the consideration of the functional integral, classical configurations corresponding to both topologies must be included [2]. The extremality condition can subsequently be imposed on the averages that result from the functional integration. It has been customary, following [3], to use a grand canonical ensemble. Here the temperature and the potential for the charges are supposed to be specified as inputs, and the average mass M and charges Q of the black hole are outputs. So the actual definition of extremality that is involved here for a Reissner - Nordström black hole with one kind of charge is $Q = M$. This may be described as *extremalization after quantization*, as opposed to the usual approach of *quantization after extremalization*. It was shown in [2] that extremalization after quantization does lead to an entropy equal to a quarter of the area. But does the approach of quantization after extremalization lead to zero entropy? Even that is not quite true [4] for the extremal Reissner - Nordström black hole: the reason is that the semiclassical approximation fails because the action does not have a stable minimum there. However, if an asymptotically anti-de Sitter version of the extremal Reissner - Nordström black hole is considered, a stable minimum does occur [5]. Consequently, there is a sensible semiclassical approximation, and as expected in [1], the entropy vanishes if quantization is carried out after extremalization. On the other hand, if quantization is carried out first, the entropy is once again given by a quarter of the area. Thus, in an asymptotically anti-de Sitter spacetime, two different kinds of extremal charged black holes exist: those obtained by quantization after extremalization, and those obtained by reversing the order of these operations. This has been confirmed in a hamiltonian framework [6].

The conclusions of [5] were reached in a grand canonical treatment. We shall go on to do a similar analysis here in the canonical ensemble. There are two motivations for this. First, it is known that the different thermodynamical ensembles are not exactly equivalent and may not lead to the same conclusions as they correspond to different physical situations. Secondly, the

canonical analysis in this system allows the consideration of the kind of transition discussed in [7] between neutral black holes in anti-de Sitter spacetime and pure anti-de Sitter spacetime. If charged black holes are envisaged, they can be imagined to decay into other charged configurations [8]. Extremal black holes defined by quantization after extremalization are well suited for this rôle because, as in the case of pure anti-de Sitter spacetime, the euclidean time coordinate here can be given an arbitrary periodicity. The same cannot be said about extremal black holes defined by quantization before extremalization, of course.

The Reissner - Nordström black hole solution of Einstein's equations in free space with a negative cosmological constant $\Lambda = -\frac{3}{l^2}$ is given by

$$ds^2 = -hdt^2 + h^{-1}dr^2 + r^2 d\Omega^2,\ A = \frac{Q}{r}dt, \tag{1}$$

with

$$h = 1 - \frac{r_+}{r} - \frac{r_+^3}{l^2 r} - \frac{Q^2}{r_+ r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}. \tag{2}$$

The asymptotic form of this spacetime is anti-de Sitter. There is an outer horizon located at $r = r_+$. The mass of the black hole is given by

$$M = \frac{1}{2}(r_+ + \frac{r_+^3}{l^2} + \frac{Q^2}{r_+}). \tag{3}$$

It satisfies the laws of black hole thermodynamics with a temperature

$$T_H = \frac{1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2}}{4\pi r_+} \tag{4}$$

and a potential

$$\phi = \frac{Q}{r_+}. \tag{5}$$

In general r_+, Q are independent, but in the extremal case they get related:

$$1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2} = 0. \tag{6}$$

The usual action for the euclidean version of the anti-de Sitter Reissner - Nordström black hole on a four dimensional manifold $\mathcal{M}$ with a boundary is given by

$$\begin{aligned} I &= -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}(R-2\Lambda) + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{\gamma}(K-K_0) \\ &+ \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}. \end{aligned} \tag{7}$$

Here γ is the induced metric on the boundary $\partial\mathcal{M}$ and K the extrinsic curvature of the boundary. K_0 is to be chosen to make the action finite. We shall study the action for off-shell configurations near the black hole solution. For simplicity, only a class of spherically symmetric metrics [3] is considered on $\mathcal{M}$:

$$ds^2 = b^2 d\tau^2 + \alpha^2 dr^2 + r^2 d\Omega^2, \tag{8}$$

with the variable r ranging between r_+ (the horizon) and r_B (the boundary), and b, α functions of r only. There are boundary conditions as usual [3, 2, 5]:

$$b(r_+) = 0, \;\; 2\pi b(r_B) = \beta. \tag{9}$$

This corresponds to the convention of fixing the range of integration of the euclidean time τ to be 2π. β is the inverse temperature at the boundary of radius r_B. There is another boundary condition involving $b'(r_+)$: It reflects the extremal/non-extremal nature of the black hole and is therefore different for the two cases:

$$\begin{aligned} \frac{b'(r_+)}{\alpha(r_+)} &= 1 \text{ in non} - \text{extremal case}, \\ &\text{and} \quad 0 \text{ in extremal case}. \end{aligned} \tag{10}$$

In the spherically symmetric situation, the vector potential was taken to be zero (a radial component may be gauged away) and the scalar potential required to satisfy the boundary conditions

$$A_\tau(r_+) = 0, \;\; A_\tau(r_B) = \frac{\beta\phi}{2\pi i}. \tag{11}$$

The boundary condition at r_B fixes the potential there and thus corresponds to the choice of the grand canonical ensemble. To go to the canonical ensemble, as we propose to do here, the electric field (or enclosed charge) rather

than the potential has to be fixed at r_B. This involves altering the action [3] with a boundary term so that the appropriate variational principle yields the equations of motion. The action (7) with the above metric takes the form:

$$\begin{aligned} I &= \frac{1}{2}\int_0^{2\pi} d\tau \int_{r_+}^{r_B} dr\Big(-\frac{2rb'}{\alpha} - \frac{b}{\alpha} - \alpha b + \Lambda\alpha b r^2\Big) - \frac{1}{2}\int_0^{2\pi} d\tau \Big[\frac{(br^2)'}{\alpha}\Big]_{r=r_+} \\ &+ I_0 + \frac{1}{2}\int_0^{2\pi} d\tau \int_{r_+}^{r_B} dr \frac{r^2}{\alpha b} A_\tau'^2. \end{aligned} \tag{12}$$

I_0 is the contribution of the K_0 term in the action and has to be chosen so as to make the action finite in the limit of large r_B with β appropriately scaled. The conversion of this action to the form appropriate to the canonical ensemble simply requires the addition of a piece $-2\pi\frac{r^2}{\alpha b}A'_\tau A_\tau|_{r_B}$. Variation of this modified action with the functions $b(r), \alpha(r)$ and $A_\tau(r)$ under boundary conditions appropriate to the new situation leads to reduced versions of the Einstein - Maxwell equations. The solution of a subset of these equations, namely the Gauss law and the hamiltonian constraint, is given by [3, 9]

$$\frac{1}{\alpha} = \Big(1 - \frac{r_+}{r} - \frac{r_+^3}{l^2 r} - \frac{q^2}{r_+ r} + \frac{q^2}{r^2} + \frac{r^2}{l^2}\Big)^{1/2}, \quad A'_\tau = -\frac{iqb\alpha}{r^2}, \tag{13}$$

with r_+ and q arbitrary at this stage. The value of q has to be fixed to define the canonical ensemble and the potential is not to be treated as being specified at the boundary in this ensemble. The above action with the boundary term added may be expressed in terms of r_+ and then has to be extremized with respect to r_+ as in [3]. The value of the action is

$$\begin{aligned} I &= -\beta r_B\sqrt{1 - \frac{r_+}{r_B} - \frac{r_+^3}{l^2 r_B} - \frac{q^2}{r_+ r_B} + \frac{q^2}{r_B^2} + \frac{r_B^2}{l^2}} + I_0 - \pi r_+^2 \ (\mathrm{non-ext\ bc}), \\ \text{and} &\ -\beta r_B\sqrt{1 - \frac{r_+}{r_B} - \frac{r_+^3}{l^2 r_B} - \frac{q^2}{r_+ r_B} + \frac{q^2}{r_B^2} + \frac{r_B^2}{l^2}} + I_0 \ (\mathrm{ext\ bc}). \end{aligned} \tag{14}$$

The first line is analogous to [3, 9], where the non-extremal condition was used in connection with a semiclassically quantized non-extremal black hole. The second line is similar to the consequence of the extremal condition used in connection with a semiclassically quantized extremal black hole [4, 2, 5]. The only difference of (14) with the corresponding equation in [5] is the

absence of a $q\beta\phi$ term which got removed when a surface term was added to alter the nature of boundary data from the grand canonical type to the canonical type.

The above "reduced action" has to be extremized with respect to r_+ in order to impose the equations of motion ignored so far.

We shall consider several cases:

1) The extremization of (14) with respect to r_+ in the *non-extremal case* yields the relation

$$\frac{\beta}{\sqrt{1-\frac{r_+}{r_B}-\frac{r_+^3}{l^2 r_B}-\frac{q^2}{r_+ r_B}+\frac{q^2}{r_B^2}+\frac{r_B^2}{l^2}}}=\frac{4\pi r_+}{(1-\frac{q^2}{r_+^2}+\frac{3r_+^2}{l^2})}. \tag{15}$$

This relation can be considered to fix r_+ in terms of the specified value of β; conversely, it also shows the expected form of β as a function of r_+. It may be noted that the value of r_+ with a given value of the left hand side ($\frac{\beta l}{r_B}$ for large r_B) is unique only for large $|q|(>\frac{l}{6})$. For smaller $|q|$, the equation for r_+ has three positive solutions for certain values of β. The second derivative of the action with respect to r_+ does not in general have a definite sign, being equal, for large r_B, to

$$\frac{2\pi(\frac{3r_+^2}{l^2}+\frac{3q^2}{r_+^2}-1)}{1-\frac{q^2}{l^2}+\frac{3r_+^2}{l^2}}. \tag{16}$$

The numerator can be guaranteed to be positive for large $|q|(>\frac{l}{6})$ only. For smaller $|q|$, some non-extremal black hole solutions may not be minima of the classical action. The entropy corresponding to the saturation of the partition function by an extremum can be confirmed to be

$$\begin{aligned}
S &= \beta^2\frac{d(I/\beta)}{d\beta}\\
&= \beta^2\frac{d}{d\beta}\Big(-r_B\sqrt{1-\frac{r_+}{r_B}-\frac{r_+^3}{l^2 r_B}-\frac{q^2}{r_+ r_B}+\frac{q^2}{r_B^2}+\frac{r_B^2}{l^2}}+\frac{I_0}{\beta}-\frac{\pi r_+^2}{\beta}\Big)\\
&= \beta^2(\frac{\pi r_+^2}{\beta^2}-\frac{I_0}{\beta^2})+\beta\frac{dI_0}{d\beta}
\end{aligned}$$

$$
\begin{aligned}
- & \quad \beta^2 \frac{dr_+}{d\beta}\frac{d}{dr_+}\Big(- r_B\sqrt{1-\frac{r_+}{r_B}-\frac{r_+^3}{l^2 r_B}-\frac{q^2}{r_+ r_B}+\frac{q^2}{r_B^2}+\frac{r_B^2}{l^2}}-\frac{\pi r_+^2}{\beta}\Big) \\
= & \quad \pi r_+^2, \qquad (17)
\end{aligned}
$$

where the I_0 terms cancel because of linear homogeneity in β and the r_+ derivative vanishes because it defines the extremum of the action. Thus the area formula is valid here.

2) The extremization of (14) can be done for the *extremal* condition, where, however, the action is homogeneous in β, which disappears from the relation fixing r_+. This is not surprising: in the extremal case q, r_+ are known to be related to each other by (6), and the temperature is arbitrary as there is no conical singularity [1]. The second derivative of the action with respect to r_+ is proportional to $\frac{3r_+^2}{l^2}+\frac{q^2}{r_+^2}$, which is positive definite, so the extremal black hole solutions are strict minima of the classical action. The grand canonical calculation led to a similar result [5] for large r_B and finite l. The canonical result holds for all r_B and persists in the limit $l \to \infty$. The entropy corresponding to the saturation of the action by this minimum is zero. This follows from the fact [1] that the action continues to be proportional to β after the extremizing value of r_+ is plugged in. Hence,

$$
S = \beta^2 \frac{d(I/\beta)}{d\beta} = 0. \qquad (18)
$$

Thus this case refers to an extremal black hole of *arbitrary* temperature and zero entropy.

3) The previous case refers to the quantized extremal black hole. As in [2, 5], there is a possibility of quantizing the black hole *before* extremizing it, *i.e.,* the two topologies may be summed over in the functional integral and the extremality condition imposed afterwards on the averaged quantities. The partition function is of the form

$$
\sum_{\text{topologies}} \int d\mu(r_+) e^{-I(r_+,\text{topology})}, \qquad (19)
$$

with I given by (14) as appropriate for non-extremal/extremal topology. The semiclassical approximation involves replacing the double integral by

the maximum value of the integrand, *i.e.,* by the exponential of the negative of the minimum I. One has to consider the variation of I as r_+ varies in both topologies. It is clear from (14) that the non-extremal action can be made lower than the extremal one because of the extra term $-\pi r_+^2$. Consequently, the partition function is to be approximated by $e^{-I_{min}}$, where I_{min} is the classical action for the *non-extremal* case, *minimized* with respect to r_+. As in the non-extremal case, this leads to an entropy equal to a quarter of the horizon area. Extremality is imposed eventually through the condition (6) on r_+. The two requirements on r_+ become consistent only in the limit $\beta \to \infty$. Thus this case refers to an extremal black hole of zero temperature and entropy equal to a quarter of the area of the horizon. It corresponds to a different way of formulating the extremal black hole from the previous case. This is the approach of *quantization before extremalization* whereas the earlier one was the pure extremal approach: *extremalization before quantization*.

4) The previous case involved a comparison of non-extremal and extremal configurations geared towards the definition of extremal black holes. One can make a more direct comparison of the actions corresponding to the first two cases. If we denote by r_+ the radius of the horizon of the non-extremal black hole as in case 1 above and refer to the corresponding quantity for the extremal black hole as in case 2 above by

$$r_0 \equiv l\sqrt{\frac{\sqrt{1+\frac{12q^2}{l^2}}-1}{6}}, \tag{20}$$

we see that

$$\begin{aligned} I_{\text{non-ex}} - I_{\text{ex}} &= -\beta r_B\sqrt{1-\frac{r_+}{r_B}-\frac{r_+^3}{l^2 r_B}-\frac{q^2}{r_+ r_B}+\frac{q^2}{r_B^2}+\frac{r_B^2}{l^2}} - \pi r_+^2 \\ &\quad +\beta r_B\sqrt{1-\frac{r_0}{r_B}-\frac{r_0^3}{l^2 r_B}-\frac{q^2}{r_0 r_B}+\frac{q^2}{r_B^2}+\frac{r_B^2}{l^2}}. \end{aligned} \tag{21}$$

The I_0 terms, which depend only on β and r_B have been cancelled out here. For large r_B, by making use of (15), we get

$$I_{\text{non-ex}} - I_{\text{ex}} = \frac{4\pi r_+}{1-\frac{q^2}{r_+^2}+\frac{3r_+^2}{l^2}}\left(\frac{r_+}{2}+\frac{q^2}{2r_+}+\frac{r_+^3}{2l^2}-\frac{r_0}{2}-\frac{q^2}{2r_0}-\frac{r_0^3}{2l^2}\right)-\pi r_+^2, \tag{22}$$

which can be recognized to be the difference of the free energies of the two black holes if one remembers the expression (3) for the mass of a black hole and the fact that the extremal black hole being considered here is of the pure type with zero entropy. On simplification,

$$I_{\text{non-ex}} - I_{\text{ex}} = \frac{\pi(r_+ - r_0)}{1 - \frac{q^2}{r_+^2} + \frac{3r_+^2}{l^2}} \left(r_+ - 3r_0 - \frac{r_+^3 + r_+^2 r_0 + r_+ r_0^2 + 9r_0^3}{l^2} \right). \tag{23}$$

For a black hole with positive temperature, $r_+ > r_0$, so that the sign of this difference depends on the last factor involving cubics in r_+ and r_0. For large enough r_0, this expression is negative for all allowed r_+, which means that all non-extremal black holes are stable against decay into the extremal black hole. However, for small r_0, *i.e.,* for small charge, there exists a range of values of r_+ for which the factor is positive, corresponding to the occurrence of non-extremal black holes capable of decaying to extremal black holes. The transition to this small charge behaviour from the large charge behaviour occurs at the positive real root of

$$516r_0^6 + 392r_0^4 l^2 + 77r_0^2 l^4 - l^6 = 0, \tag{24}$$

which is approximately given by

$$r_0 \approx 0.1105l, \tag{25}$$

corresponding to a charge of

$$|q_0| \approx 0.1125l < \frac{l}{6}. \tag{26}$$

In conclusion, we have found that the results of the grand canonical calculations of [5] are mostly reproduced in the canonical ensemble: extremal black holes in an asymptotically anti-de Sitter spacetime can be defined in two ways, one having zero entropy and arbitrary temperature and the other having zero temperature and finite entropy. However, while a non-extremal black hole in such spacetimes usually has lower free energy, in some special cases of small size it has higher free energy and can decay into extremal black holes.

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